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GEOMETRIC INTERPRETATION OF DANTZIG'S CONVEX PROGRAMMING ALGORITHM

by

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I. Review of the Algorithm [1]

We consider the following program:

$$\text{(1)} \qquad \text{PI} \begin{cases} \text{Find } x \in R \text{ such that } \phi_0(x) \text{ is maximum} \\ \text{subject to } \phi_i(x) \geq 0 \text{ for } i = 1,2,\ldots,m . \end{cases}$$

where ϕ_0 , ϕ_i are concave in x and R is compact and convex.

We replace PI by a linear program PII-K which gives a lower approximate solution $\phi_0(x)$ for PI. This linear program is constructed from a finite number of points $x = X^j$, $j \in K$, which belong to R in the following way:

$$\text{PII-K} \begin{cases} \text{Find } \lambda_j \geq 0 , \ j \in K \text{ such that} \\ \sum_{j \in K} \phi_0(X^j)\lambda_j \quad \text{is maximum} \\ \text{subject to} \\ \text{(2)} \quad \sum_{j \in K} \phi_i(X^j)\lambda_j \geq 0 \qquad i = 1,2,\ldots,m . \\ \text{(3)} \quad \sum_{j \in K} \lambda_j = 1 \end{cases}$$

Let ϕ be a column vector whose components are ϕ_i , $i = 1,2,\ldots,m$.

$A^K = \{\phi(X^j) \mid j \in K\}$, matrix of columns $\phi(X^j)$

$f^K = \{\phi_0(X^j) \mid j \in K\}$, row vector of components $\phi_0(X^j)$

$e^K = (1,1,\ldots,1)$, row vector of components $e^j = 1$, $j \in K$.

The linear program PII-K and the corresponding Kuhn-Tucker conditions in matrix notation read:

$$\text{PII-K}\quad\left|\begin{array}{ll} & \text{Find } \lambda_K \geq 0 \text{ and maximize } f^K\lambda_K \\ & \text{subject to} \\ (2) & \quad A^K\lambda_K \geq 0 \\ (3) & \quad e^K\lambda_K = 1 \end{array}\right.$$

$$\text{Kuhn-Tucker conditions}\quad\left\{\begin{array}{lllll} (4) & \Pi \quad (\Pi^O) & & \geq & 0 \\ (5) & \Pi A^K & + \Pi^O e^K + f^K & \leq & 0 \\ (6) & \Pi A^K\lambda_K & & = & 0 \\ (7) & [\Pi A^K & + \Pi^O e^K + f^K]\ \lambda_K & = & 0 \end{array}\right.$$

The dual variables Π(vector) and Π^O(scalar) correspond respectively to the relations (2) and (3).

The optimization of the linear program PII-K yields an optimal solution $\bar{\lambda}_K$, at finite distance, and corresponding dual variables $\bar{\Pi}$ and $\bar{\Pi}^O$ as functions of K .

One solves then the following auxiliary program

$$\text{PIII-K}\quad\left|\begin{array}{l} \text{Find } x \in R \text{ such that} \\ \displaystyle\sum_{i=1}^{m} \bar{\Pi}^i \phi_i(x) + \phi_O(x) \text{ is maximum} \end{array}\right.$$

Let $x = X^s$ be the optimal solution of PIII-K . The index s is then added to the set K of the indices which define the linear program PII-K .

PII-K has then one more variable λ_s . If this new variable is a candidate for entering the basis, then PII-(K+s) can be improved, if not then PII-K yields the optimal solution $\hat{x}$ for PI , viz:

(8) $$\hat{x} = \sum_{j \in K} x^j \bar{\lambda}_j$$

or with obvious notations

(9) $$\hat{x} = x^K \bar{\lambda}_K \ .$$

If the algorithm is infinite, and with the restriction of a nondegenerate solution to the initial PII, the solutions of PII-K converge to the optimal solution of PI.

II. Geometric Interpretation of PII

If we consider the point $x(\lambda_K)$ defined by

(10) $$x(\lambda_K) = x^K \lambda_K$$

where x^K has the same meaning that in (9), we have for all feasible solutions λ_K of PII:

(11) $$\phi[x(\lambda_K)] \geq A^K \lambda_K \geq 0$$

(12) $$\phi_0[x(\lambda_K)] \geq f^K \lambda_K \ .$$

In fact, $A^K \lambda_K$ and $f^K \lambda_K$ represent barycentric interpolations of concave functions. From relation (11) we know that to every feasible solution λ_K of PII-K corresponds a feasible solution $x(\lambda_K)$ to PI, i.e., that the domain of the solution $x(\lambda_K)$ obtained from feasible solutions λ_K for PII is contained in the domain of feasible solutions of PI. Moreover, relation (12) shows that the value of the objective function

of PII-K is less than or equal to the value of the objective function PI at every corresponding point of the domain on which f^K is defined. These remarks show that the linear program PII-K is a lower approximation to the given problem and this for a pair of reasons.

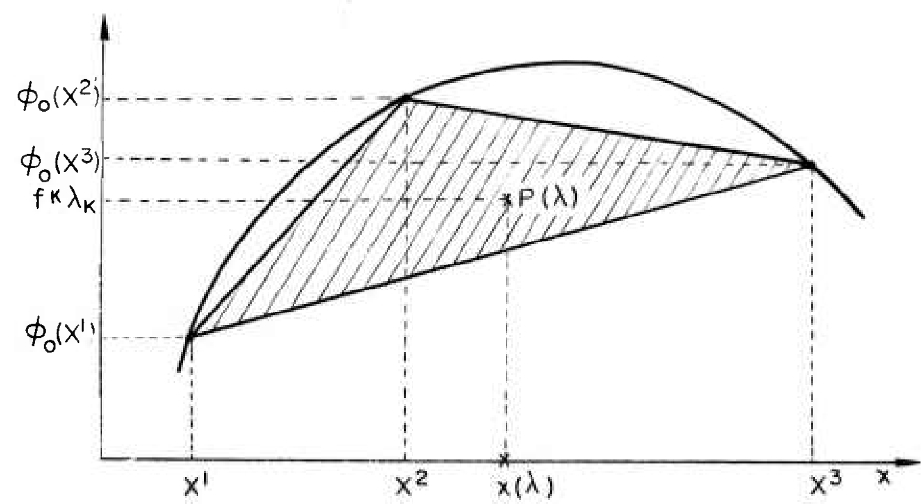


Fig. 1

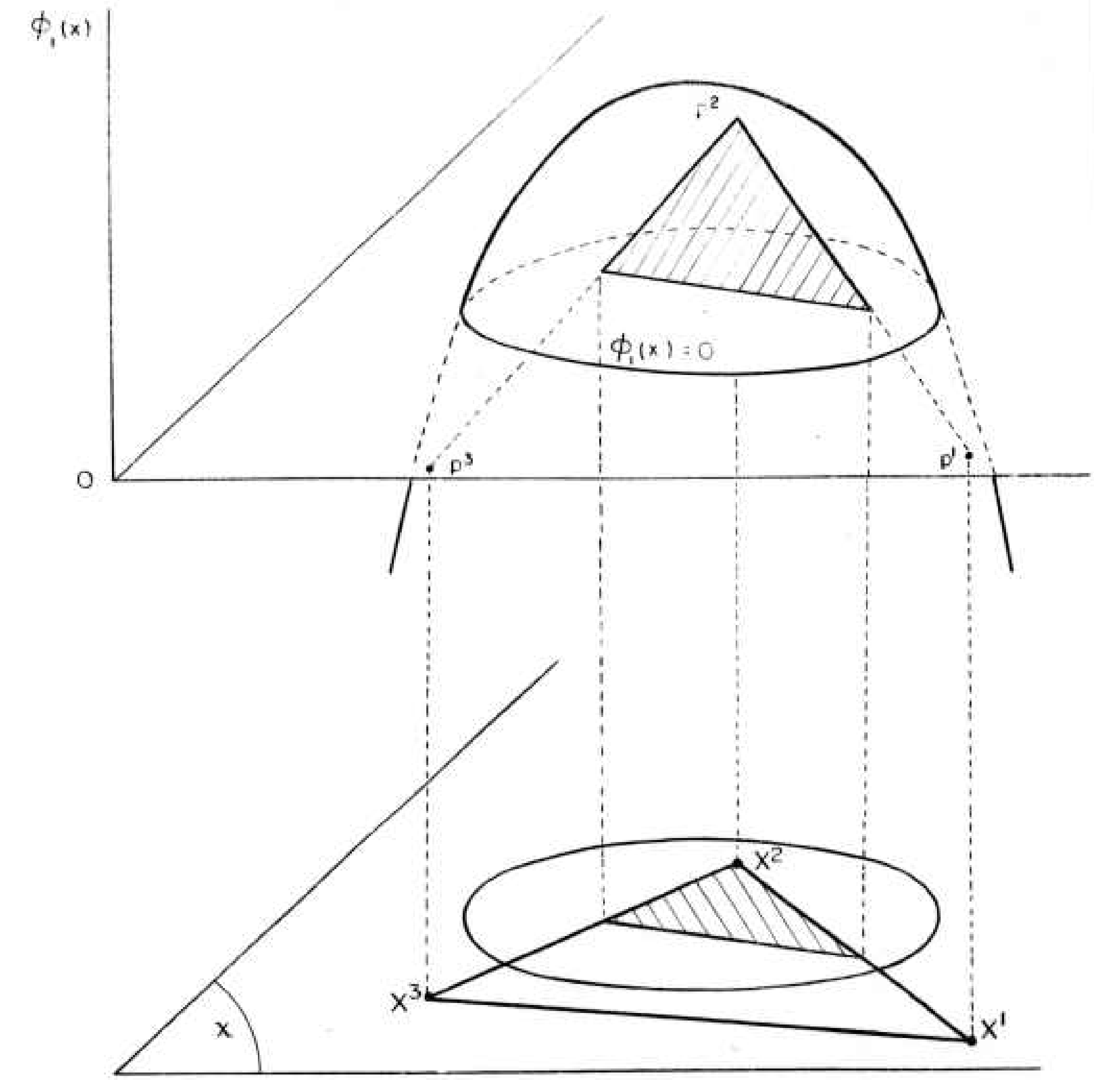


Fig. 2

III. Geometric Interpretation of PIII

PIII has the following form:

$$\text{PIII} \quad \begin{cases} \text{Find } x \in R \text{ such that } \Pi\phi(x) + \phi_0(x) \text{ is maximum} \\ \text{where } \Pi \geq 0 \text{ is a given vector.} \end{cases}$$

We have seen that PIII is used to determine the candidate variable of the simplex method to enter the basis of PII (linear program in λ).

In the standard simplex method, each step corresponds to an extreme point. To each edge leaving that vertex corresponds a non-basic variable λ_j. We then determine an edge leaving that vertex which gives the greatest increasing slope of the objective function. The slope being taken with respect to the nonbasic variable λ_j corresponding to that slope. We can also select any edge yielding a strictly positive slope (not necessarily the largest).

Here the problem is different: we do not have these edges, but we have to construct one, with positive slope (or the largest positive slope), increasing the λ-space by one dimension. The position of the new edge and the position of the new gradient to the objective function are related and are functions of the selection of a point $x = X^s$ of R. On the contrary, the projections of the normals on the new faces of PII as well as that of the new gradient in the old λ-space, remain constant.

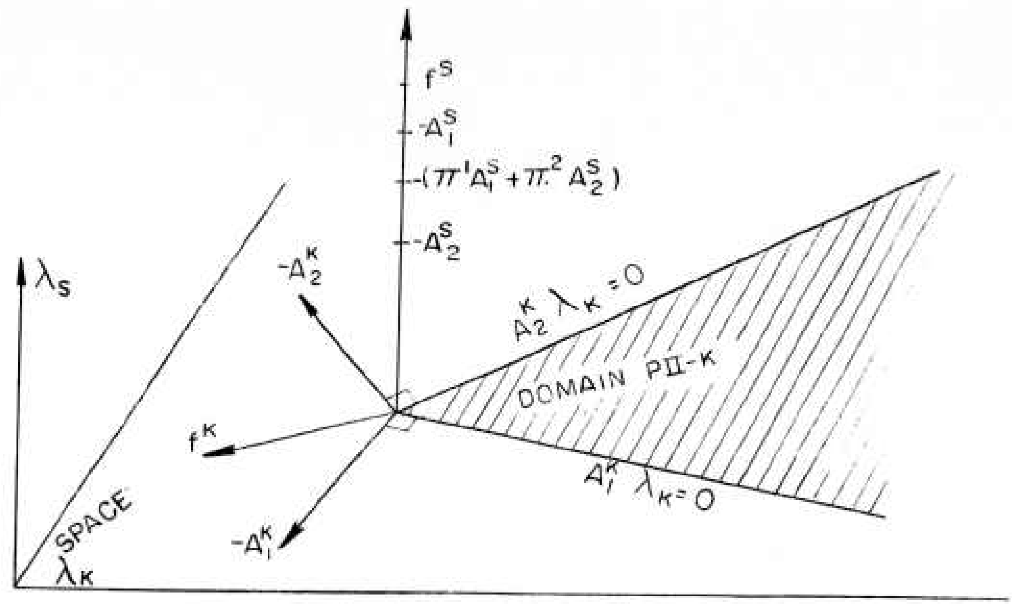


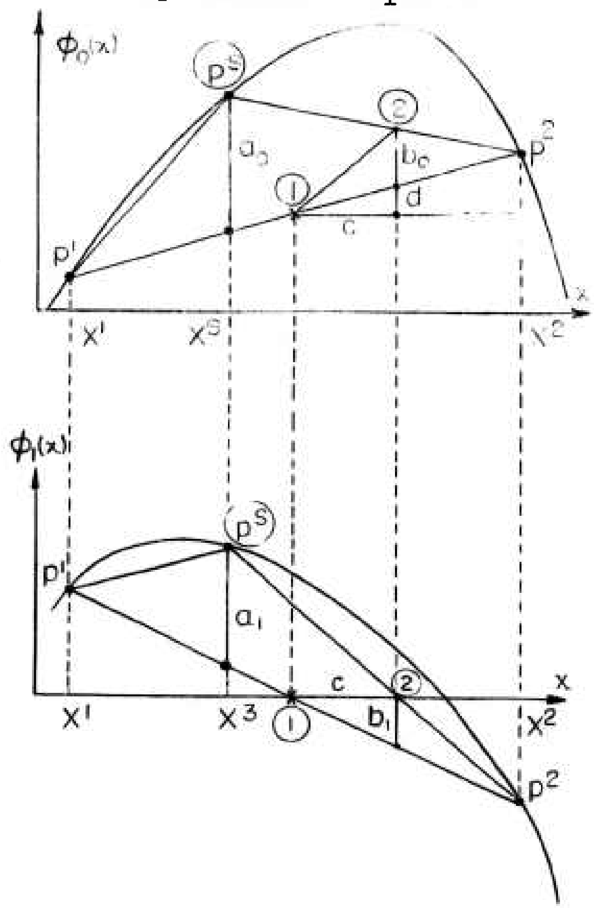
Fig. 3

When PIII-K has generated a candidate λ_s , <u>the first iteration of PII-(K+s) is the introduction of</u> λ_s <u>in the basis</u>, because it is the only candidate. But, once the basis is changed, there may exist other candidates, namely slack variables. Note that PIII-K yields an additional variable and candidate for entering the basis to PII-K , the criterion of choice of that candidate takes only into account the relative gain (or slope, as we said above) of the first iteration of PII-(K+s) and not the total gain of the following iterations necessary for the optimization of PII-(K+s) .

The optimization of PIII-K does not necessarily yield the "best" candidate for PII-(K+s), in the sense of improvement for the objective function of PII-(K+s). Let us illustrate this by an elementary example.

EXAMPLE Let x be a scalar, M a very large scalar.

$$\text{PI}\begin{cases}\text{Find } x \in [-M,M] \text{ such that } \phi_0(x) \text{ is maximum}\\[2ex] \text{subject to } \quad \phi_1(x) \geq 0\end{cases}$$



Cycle 0: $X^K = [X^1 , X^2]$
the initial basis of PII-K has two elements

Optimum PII-K

Point ① $\begin{cases}\lambda_1 & > 0\\ \lambda_2 & > 0\\ \text{slack} & = 0\end{cases}$

Optimum PIII-K
$\bar{x} = X^s$

1. Change of basis of PII-(K+s):
   Cycle 1
   ① → ②
   λ_s enters the basis
   λ_1 leaves the basis
   $\lambda_2 > 0$
   slack = 0

2. Change of basis of PII-(K+s):
   Cycle 2
   ② → P^s
   λ_2 leaves the basis
   the slack enters the basis
   $\lambda_s = 1$

Fig. 4

The slope of increase of the objective function of PII-(K+s) when we go from ① to ② (first iteration), is with respect to λ_s, we have (for notation, see Figure 4)

(13) $$p = \frac{d + b_0}{\lambda_s} .$$

We have also $d = \Pi^1 b_1$ (classical result of the parametrization of the constant column of a linear program)

$$(14) \qquad \frac{b_0}{a_0} = \frac{\lambda_s}{\lambda_s + \lambda_2} = \lambda_s \qquad \text{(similar triangles)}$$

$$(15) \qquad \frac{b_1}{a_1} = \frac{\lambda_s}{\lambda_s + \lambda_2} = \lambda_s \qquad \text{(similar triangles)} .$$

It follows that the value of the slope is:

$$(16) \qquad p = \frac{d}{\lambda_s} + \frac{b_0}{\lambda_s} = \Pi^1 a_1 + a_0 .$$

This slope is a function, by a_1 and a_0, of $x = X^s$ given by PIII-K. Letting $\bar{x}$ be the abscissa of the point ①:

$$(17) \qquad a_1(x) = \phi_1(x) - \frac{b_1}{c}(x - \bar{x}) + 0$$

$$(18) \qquad a_0(x) = \phi_0(x) + \frac{d}{c}(x - \bar{x}) + C$$

It follows

$$(19) \qquad p(x) = \Pi^1 a_1(x) + a_0(x) = \Pi^1 \phi_1(x) + \phi_0(\frac{d}{c} - \frac{d}{c})(x - \bar{x})$$
$$= \Pi\, \phi_1(x) + \phi_0(x) + C$$

We see that for the objective function of PIII the part $\Pi^1 \phi_1(x)$ has for origin the term d/λ_s, i.e., the increase of the optimal value of the objective function of PII due to the enlargement of the domain of feasible solutions, whereas the part $\phi_0(x)$ has the term b_0/λ_s for origin, corresponding to a better approximation of the function $\phi_0(x)$ of PI for the new form of the objective function of PII.

These results can be reproduced for the n-dimensional case.

Let: $\bar{\lambda}_K$ be an optimal solution of PII-K

$\bar{\lambda}'_K$ the solution obtained after the first iteration of PII-(K+s)

I the optimal basis of PII-K, with respect to $\bar{\lambda}_K$

E the set of indices of the constraints (2) of PII-K satisfied exactly (equality) for $\lambda = \bar{\lambda}_K$

$B = \begin{pmatrix} A_E^I \\ \\ e^I \end{pmatrix}$ basic matrix of PII-K

$\bar{x} = X^K \bar{\lambda}_K$

$\bar{x}' = (X^K, X^s)\, \bar{\lambda}'_K$.

We have: Variation in the objective function of PII from $\bar{x}$ to $\bar{x}'$

= Variation of the old objective function of PII from $\bar{x}$ to $\bar{x}'$

+ Variation at the point $\bar{x}'$ when we replace the old objective function by the new one.

More precisely:

<u>Variation of the old function from $\bar{x}$ to $\bar{x}'$</u> =

$$
\begin{aligned}
(20) \qquad \Delta(f^K \bar{\lambda}_K) &= f^I \Delta \bar{\lambda}_I \\
&= f^I (\bar{\lambda}'_I - \bar{\lambda}_I) \\
&= -f^I B^{-1} \begin{pmatrix} A^s \\ 1 \end{pmatrix} \bar{\lambda}'_s \\
&= (\Pi, \Pi^0) \begin{pmatrix} A^s \\ 1 \end{pmatrix} \bar{\lambda}'_s \\
&= (\Pi A^s + \Pi^0) \lambda'_s
\end{aligned}
$$

Variation at $\bar{x}'$ due to the change in functions

(21) $$f^{K+s}\,\bar{\lambda}'_{K+s} - f^{K}\,\bar{\lambda}'_{K} = f^{s}\,\bar{\lambda}'_{s}\ .$$

The total variation is

(22) $$(\Pi A^{s} + \Pi^{0} + f^{s})\,\bar{\lambda}'_{s} = [\Pi\phi(x) + \Pi^{0} + \phi_{0}(x)]\lambda'_{s}$$

and the slope, with respect to λ'_{s}, is the well-known result

(23) $$\Pi\phi(x) + \Pi^{0} + \phi_{0}(x)\ .$$

The two first terms represent the effect of the enlargement of the domain of PII-K, and the last one the improvement in the approximation of the objective function of PI.

REFERENCES

1. Dantzig, G. B., Linear Programming and Extensions, Chapter 24, Princeton University Press, Princeton, N. J., 1963.

A TRANSLATION

by Ellis Johnson and Mostafa El-Agizy

The following paper was presented at the Second International Conference on Operational Research, Aix-en-Provence, France, 1960, and may be found in its original form in the Proceedings of the Conference, published by English Universities Press, Ltd., London, (1961).

It was felt that these results were deserving of wider dissemination among "chercheurs," and this translation is a modest attempt to further this objective. The translation was done by graduate students, supervised by the undersigned; no claim of accuracy is made, but it is hoped that at least the spirit of the original is maintained.

Acknowledgment is due the author, who kindly gave permission for the translation, and the International Federation of Operational Research Societies, who hold the copyright.

W. S. Jewell

W. S. Jewell

THE PROBLEM OF THE MAXIMUM FLOW IN TRANSPORTATION

WITH CORRESPONDING CONSTRAINTS

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In a transportation network in which for each entry arc there corresponds an exit arc and vice versa, corresponding pairs of entries and exits are required to carry the same flow. Under these conditions, we can obtain the maximum flow by applying the Ford-Fulkerson algorithm to a sequence of networks, each of which is obtained from the labeling of the preceding network. The final result is attained in a finite number of steps.

1. Introduction

The study of the problem of engine scheduling at the S. N. C. F. led us to the following theoretical problem:

We consider a transportation network in which there is a two-way correspondence between entry and exit arcs.

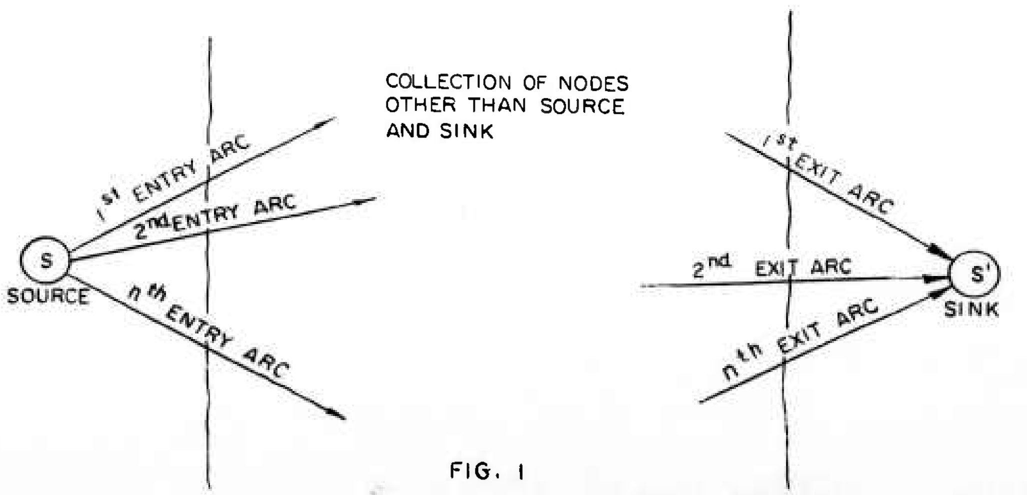


FIG. 1

To each arc u in the network is attached a capacity $C(u)$. The problem is to send flow in this network subject to the capacity constraints: $\phi(u) \leq C(u)$; the correspondence constraints: flow through an entry arc = flow through the corresponding exit arc, and such that the flow is maximum.

2. <u>The Problem of the Maximum Flow in a Transportation Network (To review known notions</u>)

We shall recall the principles of the Ford-Fulkerson algorithm which allows us to solve the following simpler problem:

Given a transportation network; for example, the one shown in Fig. 2, in which the numbers between parentheses represent arc capacities, the problem is how to send maximum flow from source to sink.

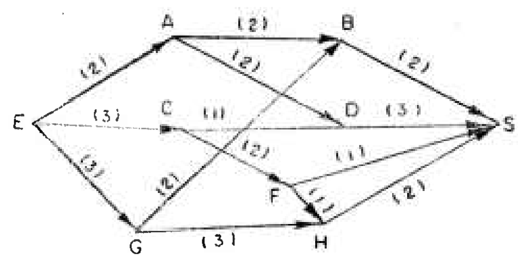


FIG. 2

The algorithm of Ford-Fulkerson consists of the following operations:

(1) Send an arbitrary flow: as illustrated in Figure 3.

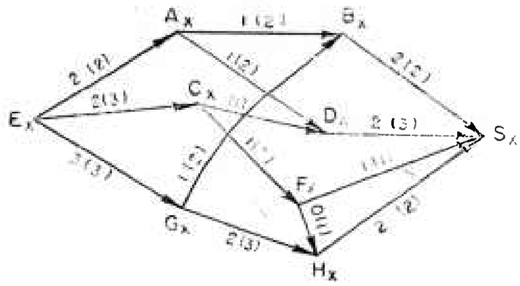


FIG. 3

(2) Proceed with labeling nodes according to the following rules:

a) S must be labeled (with X)

b) For a non-saturated arc (x , y) if x is labeled, label y

c) For a non-empty arc (y , x) if x is labeled, label y

(3) a) if S' is labeled, we can improve the flow by one unit. For example, in Figure 3, S is labeled by rule a; C , F , and G by rule b; E by rule c; B by rule b; A by rule c; D and S by rule b. If we add one unit of flow to SC, CF, FG, subtract one unit of the flow from EG, add one unit to EB, subtract one from AB, add one unit to AD and DS' we obtain, as shown in Figure 4, a one unit improvement over the previous flow.

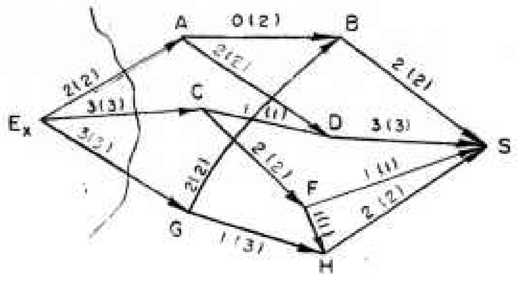


FIG. 4

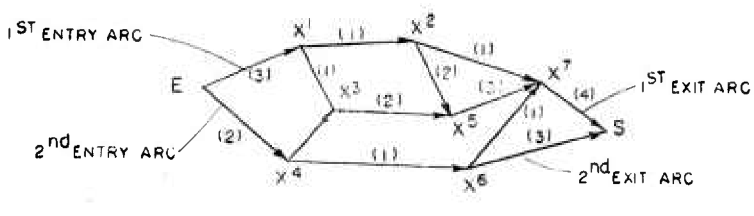
b) If it is not possible to label S' (for example in Figure 4, only S can be labeled), then maximum flow is attained. To prove this, consider the set of arcs U^+ such that the beginning nodes are labeled and the end nodes are unlabeled, and the set of arcs U^- such that the end nodes are unlabeled and the beginning nodes are labeled. The flow through the network from S to S' is also the flow out of the set of labeled nodes (containing S) to the set of unlabeled nodes (containing S'). Accordingly, it is the sum of flow in arcs going from the first set to the second minus the sum of flow going from the second to the first set

$$\sum_{u \in U^+} \phi(u) - \sum_{u \in U^-} \phi(u)$$

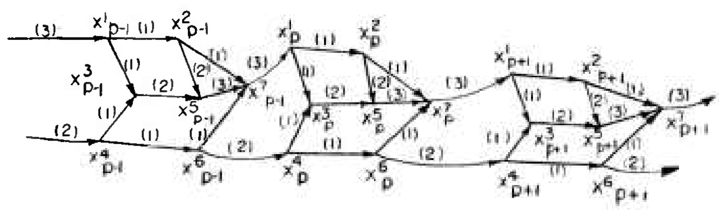
3. Return to the Main Problem: Auxiliary Graph

Let $x^1, x^2, \ldots x^m$ be the nodes other than source and sink in the network under consideration. We shall introduce a capacitated auxiliary graph G with an unlimited number of nodes denoted by x_p^i when

$i = 1, 2, \ldots m$ and where $-\infty < p < \infty$. The arcs of this graph and their capacities will be defined in the following manner:



INITIAL NETWORK



AUXILIARY GRAPH

FIG. 5

a) For each value of p, the nodes $x_p^1, x_p^2, \ldots, x_p^m$ are connected by arcs exactly like the one in the initial network.

b) For each entry or exit arc in the initial network say $\overrightarrow{Sx^i}$, $\overrightarrow{x^jS'}$, and for each p there corresponds an arc $\overrightarrow{x_p^j x_{p+1}^i}$ with

a capacity equal to the smallest of the two capacities.

Figure 5 illustrates the construction of such an auxiliary graph.

In a network, send flow which assigns to every arc u a positive integer flow $n(u) \leq C(u)$ such that at every node other than source and sink the conservation equations are satisfied; i.e., the sum of the flows in entering arcs equals the sum of the flows in exit arcs. We say likewise for the auxiliary graph G that we have sent a flow into G if to each arc u is assigned a positive integer flow $n(u) \leq C(u)$ so that at each node the conservation equations are satisfied. We say that the flow is periodic if the flow is the same in any two homologous arcs, where homologous arcs are two arcs which correspond to the same arc or pair of arcs in the initial network. Under these conditions, it turns out to be the same to send flow in the initial network obeying the correspondence constraints or sending periodic flow in the auxiliary graph G.

4. Expression for the Quantity to be Maximized

The quantity to be maximized, which in the initial network is the sum of flows in entering arcs, is found in the graph G as the sum of flows in the arcs connecting nodes with subscript p to the nodes with subscript $p + 1$. However, we can give it a more general expression. Consider a function $p(i)$ which assigns to every value of the superscript i a value of the subscript p. Each function $p(i)$ defines a partition of nodes of graph G into two sets: the set $E[p(i)]$ of x_p^i such that $p \leq p(i)$, and the complementary set. Calling $U[p(i)]$ the set of arcs for which only the origin node belongs to $E[p(i)]$, $W[p(i)]$ the set of arcs for which only the terminal node belongs to $E[p(i)]$ and finally, given a periodic flow passing through G, we call $n(u)$ the

flow in arc u . If we take for the $p(i)$ the constant function of i , say $p(i) = p_o$, then the quantity to be maximized according to what we mentioned previously is

$$\sum_{u \in U[p(i)]} n(u) ;$$

on the other hand for such a function constant in i , the set $W[p(i)]$ is empty since we did not have in the construction of G arcs with the origin node having subscript $p + 1$, and terminal node having subscript p . Accordingly, we can put the quantity to be maximized in the form

$$\sum_{u \in U[p(i)]} n(u) - \sum_{u \in W[p(i)]} n(u)$$

where $p(i)$ is a function constant in i . From the flow conservation equations, we can easily deduce that the quantity

$$\sum_{u \in U[p(i)]} n(u) - \sum_{u \in W[p(i)]} n(u)$$

does not depend on $p(i)$ and we denote this quantity by $F[n(u)]$.

If we define $C[p(i)] = \sum_{u \in U[p(i)]} C(u)$ the following theorem is obtained:

THEOREM: For any function $p(i)$, $F[n(u)] \leq C[p(i)]$. Indeed $0 \leq n(u) \leq C(u)$ and hence

$$F[n(u)] = \sum_{u \in U[p(i)]} n(u) - \sum_{u \in W[p(i)]} n(u) \leq \sum_{u \in U[p(i)]} C(u) = C[p(i)] .$$

5. Transportation Network Associated with $p(i)$

With a function $p(i)$ is associated the transportation network $G[p(i)]$ defined as follows:

The nodes of $G[p(i)]$ are the elements of $\Delta[p(i)] = E[p(i)] - E[p(i) - 1]$ to which are added four points I, J, I_o, J_o.

The arcs of $G[p(i)]$ are the arcs of G after the following transformation: (i) replace by I any node of $E[p(i) - 1]$; (ii) replace by J any node not in $E[p(i)]$; (iii) discard any arc with both ends at the same node; (iv) add the arcs (I_o, I) and (J, J_o) both with infinite capacity.

Three categories of arcs will be distinguished:

The set U_1 of arcs which do not touch I or J;

the set U_2 of arcs whose origin is I or whose end is J;

the set U_3 of arcs whose end is I or whose origin is J.

Figure 6 shows a network $G[p(i)]$ corresponding to the example of Figure 5 taking $p(1) = p(2) = p(3) = p(4) = 1$, $p(5) = p(6) = p(7) = 0$.

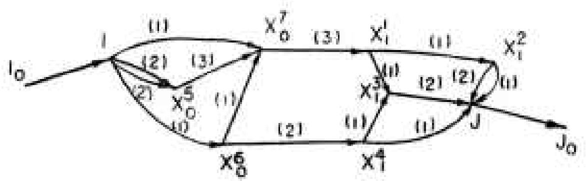


FIG 6

6. Properties of the Network Associated with $p(i)$

Let $N[p(i)]$ be the value of the maximum flow in $G[p(i)]$. Then

$N[p(i)]$ is nonnegative since a zero flow is always possible. In addition, $N[p(i)] \leq G[p(i)]$ where $C[p(i)]$ is the capacity of the arc consisting of the arcs other than J , J_o having one end at J .

The Ford-Fulkerson algorithm applied to this network yields a partition of the nodes of $G[p(i)]$ into labeled nodes (including I_o) and unlabeled nodes (including J_o) . For $p(i)$ let $\phi_{p(i)}(i)$ be a corresponding function equal to $p(i)$ if $x^i_{p(i)}$ is labeled, to $p(i) - 1$ if $x^i_{p(i)}$ is not labeled. Then $C[\phi_{p(i)}(i)]$ is precisely the capacity of the cut indicated by the preceding partition, so then $N[p(i)] = C[\phi_{p(i)}(i)]$.

7. Solution of the Given Problem

Suppose for the moment that $p(i)$ is such that $N[p(i)] = C[p(i)]$ and the flow in $G[p(i)]$ is maximum. Then the arcs of U_2 are saturated, and those of U_3 are empty.

Furthermore, an arc of U_λ and an arc of U_μ for $\lambda \neq \mu$ or for $\lambda = \mu = 1$ cannot correspond to identical or homologous arcs of G , and any arc u of G is homologous to at least one arc corresponding to an arc v of $G[p(i)]$. If $u \in U_1$, let $n(u)$ be the flow in v . If $v \in U_2$, let $n(u)$ the flow in v . If $v \in U_3$, let $n(u) = 0$.

A periodic flow in G is obtained in this way. The periodicity is trivial, and it is easily seen that the capacity constraints and conservation equations are satisfied. This flow is maximum because

$$F[n(u)] = \sum_{u \in U[p(i)]} n(u) - \sum_{u \in W[p(i)]} n(u) = \sum_{u \in U[p(i)]} C(u) - \sum_{u \in W[p(i)]} 0 = C[p(i)].$$

and a previous theorem shows that any flow in G satisfies $F[n(u)] \leq C[p(i)]$.

Hence, we can obtain the answer to the given problem once a function $p(i)$ is known such that $N[p(i)] = C[p(i)]$.

Thus, the following theorem is important:

<u>THEOREM</u>: There exists $p(i)$ such that $N[p(i)] = C[p(i)]$.

PROOF: Consider the sequence of functions $p_o(i)$, $p_1(i),\ldots,p_n(i),\ldots$ defined beginning with the arbitrary function $p_o(i)$ by the recurrence relation $p_{n+1}(i) = \phi_{p_n(i)}(i)$. To it corresponds a sequence of positive integers $C[p_o(i)]$, $C[p_1(i)],\ldots,$ $C[p_n(i)],\ldots$. This sequence cannot be strictly decreasing. Hence, there is an integer n such that $C[p_n(i)] = C[p_{n+1}(i)]$. But $N[p_n(i)] = C[\phi_{p_n(i)}(i)] = C[p_{n+1}(i)]$. Hence $N[p_n(i)] = C[p_n(i)]$.

The preceding proof defines an algorithm which will be applied to the network of Figure 5 as an example:

To begin, let us take $p_o(i) = 0$. $G[p_o(i)]$ is the initial network (except for the modifications of the capacities of entry and exit arcs). Figure 7(a) shows a maximum flow in this network (with the same flow in corresponding entry and exit arcs).

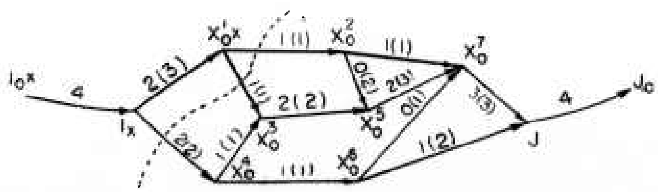


FIG. 7a

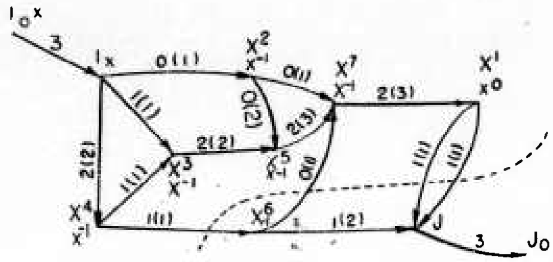


FIG 7b

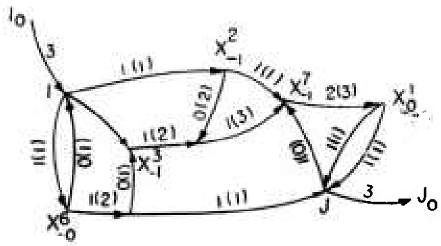


FIG 7c

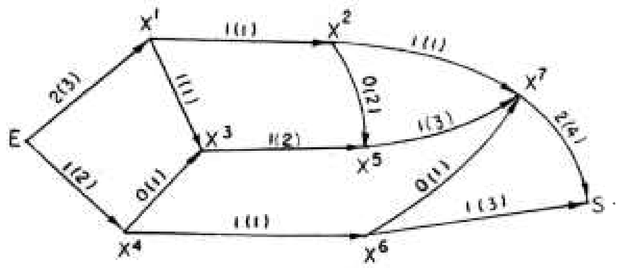


FIG 7d

Then we obtain $p_1(1) = 0$, $p_1(2) = p_1(3) = p_1(4) = p_1(5) = p_1(6) = p_1(7) = -1$ and the network of Figure 7(b) in which a maximum flow is found. From it $p_2(6) = -2$, $p_2(1) = 0$, $p_2(2) = p_2(3) = p_2(4) = p_2(5) = p_2(7) = -1$, and the network of Figure 7(c) is obtained. In figure 7(c), $N[p_2(i)] = C[p_2(i)]$.

Then we obtain the solution of the problem (Figure 7(d)).

A REVIEW

A TECHNIQUE FOR RESOLVING DEGENERACY IN LINEAR PROGRAMMING

by

Philip Wolfe
J. Soc. Indust. Appl. Math. 11 (1963), 205-211

Prepared by George B. Dantzig
for
"Mathematical Reviews"

Geometrically, the simplex method for solving linear programs passes iteratively from one extreme point to a selected neighbor in a convex polyhedral set. If the set is defined by a system of m linear equations in n non-negative variables then algebraically an extreme point solution E is obtained if, setting $n - m$ variables equal to zero, there is a unique non-negative solution in the remaining "basic" set of m-variables. If the same extreme point E corresponds to more than one basic set, the solution is called "degenerate." If so, the value of linear "objective" form (to be minimized) may not decrease from iteration to iteration. Several special selection rules for successively choosing basic sets have been proposed to guarantee, under degeneracy, termination in finite number of steps.

According to the author, his paper "is closely related to the material of Dantzig's Inductive proof of the simplex method. That paper may be viewed as showing the existence of a class of choice rules which prevent cycling in the simplex method and the present paper viewed as exhibiting a member of that class." At any iteration let R be the set of equations in the simplex tableau whose constant terms are zero and from which the pivot term can be selected. These

constant terms are each replaced by the polynomial $0 + \epsilon$ where $\epsilon > 0$. (In selecting pivots, polynomial expressions in ϵ and later $\epsilon^2, \ldots$ are compared using a lexicographic ordering of their leading coefficients). Degeneracy becomes less "deep" if a pivot term occurs outside of R; if so, there will be a positive decrease of the objective form. However, degeneracy deepens if a proper subset of R' of the R equations should develop polynomial expressions in ϵ with all zero coefficients. For these equations, the constants are replaced by $0 + 0\epsilon + \epsilon^2$ and pivots are now selected from R', etc.

The author's proof is essentially inductive (although not exactly so stated). The number of pivots selected in the subset R will be finite because, by induction, there can be only a finite number of steps in R' which results in an optimum solution or a non-zero decrease in the ϵ term of the objective form. This implies nonrepetition in the choice of basic sets of variables.

The paper concludes with some empirical observations on the number of iterations required to solve some simple degenerate problems using, unfortunately, a different pivot selection rule.

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